INTERACTIONS BETWEEN ELECTRONS AND LATTICE VIBRATIONS IN A SUPERCONDUCTOR

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A perturbation theory is developed for the Green's function in which the Green's function calculated for the superconducting ground state is used as the zero approximation. Dyson equations are written down from which the electron Green's function can be determined. Interaction between electrons and phonons is not assumed to be small. The spectrum and the damping of the excitations are calculated.

1. INTRODUCTION

As Migdal[1] has shown in his work, a perturbation series for the vertex function of electron-phonon interaction converges rapidly, independent of whether the interaction is weak or strong. The expression $\lambda_0 h\omega_0/E_F$ served as the expansion parameter, where $\lambda_0 \lesssim 1$, ω_0 — maximum frequency of the phonons, E_F — Fermi energy. For this reason, it was possible to solve the Dyson equation and find the Green's function of the electrons and the phonons with accuracy up to $\lambda_0 h\omega_0/E_F$.

Application of this method to the study of the superconducting state is not possible if one starts out from the states of non-interacting electrons. Contemporary superconductivity theory makes use of a new system of approximate wave functions.[2,3] The corresponding energy spectrum has a gap

$$\varepsilon(k) = \sqrt{\xi^2(k) + C^2(k)},$$

where $\xi(k)$ is the spectrum of the normal state reckoned from the Fermi surface and $C(k)$ is a quantity defining the gap. The wave functions of the new collection of states depend both on the $C(k)$ and on the parameter. It is possible to determine them, for example, by a variational method[2] or by the method of Bogolyubov.[3] Starting out from the new states, Bogolyubov determined the single particle spectrum of the superconductor with accuracy up to $\lambda_0 h\omega_0/E_F$.[4]

It is of interest to apply the Migdal method, using superconductivity theory, to the determination of the Green's functions of the electrons in the superconductor. In addition to the results of Bogolyubov et al.,[4] this also allows us to compute the damping of the excitation.

The Green's functions of the electrons in a superconductor were computed by Gor'kov[5] for a model with a simplified four-fermion interaction. Use of a simplified Hamiltonian assumes the averaging of all quantities over the region with linear dimensions of order $v_0/\omega_0 \sim 10^{-5}$ cm (v_0 — velocity of electrons on the Fermi surface). Since the depth of penetration of the magnetic field in a superconductor far removed from the transition point has the same order of magnitude, then the obtaining of equations of the type of the Gor'kov equations for the Fröhlich Hamiltonian is of particular interest. Such equations are derived in the present work with the aid of superconductivity theory, in which the Green's functions computed for the "superconducting" BCS-Bogolyubov ground state are used as the zero approximation.

2. ZERO HAMILTONIAN IN THE INTERACTION REPRESENTATION

Following the work of Migdal, we shall consider a system with the Fröhlich Hamiltonian (the system of units is used in which $h = m = 1$):

$$H = \int d\mathbf{x}\,\{\psi_\sigma^+(\mathbf{x})\,[H(\mathbf{x}) - \mu]\,\psi_\sigma(\mathbf{x}) + \psi_\sigma^+(\mathbf{x})\,\psi_\sigma(\mathbf{x})\,\varphi(\mathbf{x})\} + H_{ph}, \tag{1}$$

$$\psi_\sigma(\mathbf{x}) = V^{-1/2}\sum_{\mathbf{k}} a_{\mathbf{k}\sigma} e^{i\mathbf{k}\mathbf{x}},$$

$$\varphi(\mathbf{x}) = V^{-1/2}\sum_{q<q_M} \alpha_q\,(b_{\mathbf{q}} + b_{-\mathbf{q}}^+)\,e^{i\mathbf{q}\mathbf{x}}. \tag{2}$$

Here $H(\mathbf{x})$ is the one-electron Hamiltonian, q_M is the maximum momentum of the phonons; for $q \ll q_M$,

$$\alpha_q^2 = \lambda_0 \pi^2 s q / k_0, \tag{3}$$

$\lambda_0 \lesssim 1$ is the dimensionless parameter of Fröhlich, k_0 is the Fermi momentum, s is the sound velocity. The chemical potential μ is included in the Hamiltonian, since states will be used with a variable number of electrons. The explicit form of the phonon Hamiltonian is not needed.

We now introduce an abbreviated notation which is suitable for further exposition. We number the operators $\psi_{1/2}$, $\psi_{-1/2}$, $\psi^+_{-1/2}$, $\psi^+_{1/2}$ and will then deal with the "vector" ψ_α ($\alpha = 1, 2, 3, 4$). The operators ψ_α satisfy the commutation relations

$$\{\psi_\alpha(\mathbf{x}),\ \psi_\beta(\mathbf{x}')\} = I_{\alpha,\beta}\delta(\mathbf{x}-\mathbf{x}'), \tag{4}$$

$$\hat{I} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

$\hat{I}^2 = \hat{E}$ ($\hat{E}$ is the unit matrix).

Along with ψ_α, we shall consider

$$\psi^\alpha = I_{\alpha\beta}\psi_\beta, \tag{6}$$

such that

$$\{\psi^\alpha(\mathbf{x}),\ \psi_\beta(\mathbf{x}')\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'). \tag{7}$$

In this notation, the Hamiltonian has the following form:

$$H = H_{el} + H_{ph} + H_{int},$$
$$H_{el} = \int d\mathbf{x}\psi^\alpha(\mathbf{x})[H(\mathbf{x}) - \mu]\psi_\beta(\mathbf{x})N_{\beta\alpha},$$
$$H_{int} = \int d\mathbf{x}\psi^\alpha(\mathbf{x})\psi_\beta(\mathbf{x})\varphi(\mathbf{x})N_{\beta\alpha}. \tag{8}$$

The matrix $\hat{N}$ guarantees the normal order of the operators:

$$\hat{N} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

As has already been noted, application of perturbation theory to the Hamiltonian of non-interacting electrons H_{el} does not lead to superconductivity in any finite order, independently of whether we expand the Green's function or the vertex function in a series. Our aim is to construct a perturbation theory for the Green's function on the basis of the collection of states employed in contemporary superconductivity theory. Bogolyubov[3,6] determined these states as the eigenstates of the operator of the number of particles

$$\int d\mathbf{x}[\chi_0^+(\mathbf{x})\chi_0\mathbf{x} + \chi_1^+(\mathbf{x})\chi_1(\mathbf{x})],$$

where χ^+, χ are the creation and annihilation operators of the excitations, introduced by means of the canonical transformation

$$\chi_0(\mathbf{x}) = \int dy[U(\mathbf{x},\ y)\psi_{\pm 1/2}(y) - V(\mathbf{x},\ y)\psi^+_{\pm 1/2}(y)],$$
$$\chi_1(\mathbf{x}) = \int dy[U(\mathbf{x},\ y)\psi_{-1/2}(y) + V(\mathbf{x},\ y)\psi^+_{\pm 1/2}(y)]. \tag{10}$$

U and V are, generally speaking, complex functions. Such conditions should be imposed on them that the operators χ satisfy the commutation relations for Fermi operators. The ground state is determined as a vacuum excitation.

In connection with what has been pointed out, we redetermine the zero Hamiltonian and write down the total Hamiltonian in the following form:

$$H = H_0 + H', \qquad H_0 = \tilde{H}_{el} + H_{ph},$$
$$H' = H_{int} - (\tilde{H}_{el} - H_{el});$$
$$\tilde{H}_{el} = \int \tilde{H}_{\alpha\beta}(\mathbf{x},\ y)\psi^\beta(\mathbf{x})\psi_\alpha(y)\,d\mathbf{x}\,dy,$$
$$\tilde{H}_{\alpha\beta}(\mathbf{x},\ y) = \tilde{H}^*_{\beta\alpha}(y,\ \mathbf{x}). \tag{11}$$

In order that the matrix $\tilde{H}_{\alpha\beta}$ be diagonal in the χ-representation, it must have the following form in the representation of the ψ-operators:

$$\hat{\tilde{H}}(\mathbf{x},\ y) = \begin{pmatrix} \tilde{H}(\mathbf{x},\ y) & 0 & \frac{1}{2}C(\mathbf{x},\ y) & 0 \\ 0 & \tilde{H}(\mathbf{x},\ y) & 0 & -\frac{1}{2}C(\mathbf{x},\ y) \\ \frac{1}{2}C^*(y,\ \mathbf{x}) & 0 & 0 & 0 \\ 0 & -\frac{1}{2}C^*(y,\ \mathbf{x}) & 0 & 0 \end{pmatrix}. \tag{12}$$

In the absence of external fields, $\tilde{H}(x, y)$ and $C(x, y)$ are real functions of the difference in the arguments. We denote their Fourier components by $\tilde{\xi}(k)$ and $C(k)$, respectively. As we shall see, $\tilde{\xi}(k)$ will represent the renormalized energy of the electron excitations in the normal state, while $C(k)$ will determine the energy gap. The transformation functions (10) are also real if there are no external fields; their Fourier components are connected with $\tilde{\xi}(k)$ and $C(k)$ in the following fashion:

$$u_k^2 = \frac{1}{2}\left(1 + \tilde{\xi}(k)/\sqrt{\tilde{\xi}^2(k) + C^2(k)}\right),$$
$$v_k^2 = \frac{1}{2}\left(1 - \tilde{\xi}(k)/\sqrt{\tilde{\xi}^2(k) + C^2(k)}\right). \tag{13}$$

We now determine the Green's functions

$$G_\alpha^\beta(\mathbf{x}, t; \mathbf{x}', t') = -i\langle T\psi_\alpha(\mathbf{x}, t)\psi^\beta(\mathbf{x}', t')\rangle, \tag{14}$$
$$D(\mathbf{x}, t;\ \mathbf{x}', t') = -i\langle T\varphi(\mathbf{x}, t)\varphi(\mathbf{x}', t')\rangle. \tag{15}$$

For brevity, we shall call the matrix $\hat{G}$ with the

matrix elements (14) the electron Green's function. Sometimes we shall find it convenient to deal with quantities

$$G_{\alpha\beta}(\mathbf{x},t;\ \mathbf{x}',t') = -i\langle T\psi_\alpha(\mathbf{x},t)\psi_\beta(\mathbf{x}',t')\rangle.$$

The electron Green's function of zero approximation satisfies the matrix equation

$$i\frac{\partial}{\partial t}\hat{G}^{(0)}(\mathbf{x},t;\mathbf{x}',t') - \int d\mathbf{y}\,\hat{H}(\mathbf{x},\mathbf{y})\hat{G}^{(0)}(\mathbf{y},t;\ \mathbf{x}',t') = \hat{E}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'),$$

$$\hat{H}(\mathbf{x},\mathbf{y}) = \hat{\tilde{H}}(\mathbf{x},\mathbf{y}) - \hat{I}\hat{\tilde{H}}^*(\mathbf{x},\mathbf{y})\hat{I}$$

$$= \begin{pmatrix} \tilde{H}(\mathbf{x},\mathbf{y}) & 0 & C(\mathbf{x},\mathbf{y}) & 0 \\ 0 & \tilde{H}(\mathbf{x},\mathbf{y}) & 0 & -C(\mathbf{x},\mathbf{y}) \\ C^*(\mathbf{y},\mathbf{x}) & 0 & -\tilde{H}^*(\mathbf{x},\mathbf{y}) & 0 \\ 0 & -C^*(\mathbf{y},\mathbf{x}) & 0 & -\tilde{H}^*(\mathbf{x},\mathbf{y}) \end{pmatrix}. \quad (16)$$

Transforming to Fourier components in x and t, we obtain

$$[\omega\hat{E} - \hat{H}(k)]\hat{G}^{(0)}(k,\omega) = \hat{E}. \quad (17)$$

The solution of Eq. (17) can be written as follows:

$$\hat{G}^{(0)}(k,\omega) = \begin{pmatrix} G^{(0)}(k,\omega) & 0 & F^{(0)}(k,\omega) & 0 \\ 0 & G^{(0)}(k,\omega) & 0 & -F^{(0)}(k,\omega) \\ F^{(0)}(k,\omega) & 0 & -G^{(0)}(k,-\omega) & 0 \\ 0 & -F^{(0)}(k,\omega) & 0 & -G^{(0)}(k,-\omega) \end{pmatrix}$$

$$G^{(0)}(k,\omega) = \frac{u_k^2}{\omega - \varepsilon(k) + i\delta} + \frac{v_k^2}{\omega + \varepsilon(k) - i\delta}$$

$$F^{(0)}(k,\omega) = \frac{C(k)}{(\omega - \varepsilon(k) + i\delta)(\omega + \varepsilon(k) - i\delta)}$$

$$\left(\varepsilon(k) = \sqrt{\tilde{\xi}^2(k) + C^2(k)}\right). \quad (18)$$

The imaginary part was determined by a theorem of Landau.[7] The functions $G^{(0)}$ and $F^{(0)}$ have the same structure as the corresponding Gor'kov functions.[5]

To construct a perturbation theory, it is necessary to establish the rule of calculation of the averages, over the ground state of the Hamiltonian H_0, of the T-product of the ψ-operators in the interaction representation. Direct application of the Wick theorem is not possible, since the concept of a normal product does not exist for the ψ-operators. We transform to χ-operators by means of the transformation (10), which we shall write for brevity in the form $\psi_\alpha = U_{\alpha\beta}\chi_\beta$. With reference to the χ-operators, the ground state is a vacuum one. Therefore, the usual rules of calculation of the mean are applicable. Considering the T-product of four operators for simplicity, we obtain

$$\langle T\psi_\alpha\psi_\beta\psi_\gamma\psi_\delta\rangle = U_{\alpha\alpha_1}U_{\beta\beta_1}U_{\gamma\gamma_1}U_{\delta\delta_1}\langle T\chi_{\alpha_1}\chi_{\beta_1}\chi_{\gamma_1}\chi_{\delta_1}\rangle$$
$$= U_{\alpha\alpha_1}U_{\beta\beta_1}U_{\gamma\gamma_1}U_{\delta\delta_1}\{\langle T\chi_{\alpha_1}\chi_{\beta_1}\rangle\langle T\chi_{\gamma_1}\chi_{\delta_1}\rangle$$
$$-\langle T\chi_{\alpha_1}\chi_{\gamma_1}\rangle\langle T\chi_{\beta_1}\chi_{\delta_1}\rangle + \langle T\chi_{\alpha_1}\chi_{\delta_1}\rangle\langle T\chi_{\beta_1}\chi_{\gamma_1}\rangle\}$$
$$= \langle T\psi_\alpha\psi_\beta\rangle\langle T\psi_\gamma\psi_\delta\rangle$$
$$-\langle T\psi_\alpha\psi_\gamma\rangle\langle T\psi_\beta\psi_\delta\rangle + \langle T\psi_\alpha\psi_\delta\rangle\langle T\psi_\beta\psi_\gamma\rangle.$$

Here it is evident that, in the notation used, the average vacuum states are computed according to the usual rules. In this case the matrix $\hat{G}^{(0)}$ serves as the Green's function.

3. DERIVATION OF THE EQUATIONS AND CALCULATIONS OF THE ELECTRON GREEN'S FUNCTION

We begin with the equation for the Green's function

$$\left[i\frac{\partial}{\partial t}\delta_{\alpha\gamma} - H^{(0)}_{\alpha\gamma}(\mathbf{x}) + \mu\Gamma^{(0)}_{\alpha\gamma}\right]G^{\beta}_{\gamma}(\mathbf{x},\ t;\ \mathbf{x}',\ t')$$
$$= \delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')$$
$$- i\Gamma^{(0)}_{\alpha\gamma}\langle T\varphi(\mathbf{x},\ t)\,\psi_\gamma(\mathbf{x},\ t)\,\psi^\beta(\mathbf{x}',t')\rangle, \quad (19)$$

$$\hat{H}^{(0)}(\mathbf{x}) = \begin{pmatrix} H(\mathbf{x}) & 0 & 0 & 0 \\ 0 & H(\mathbf{x}) & 0 & 0 \\ 0 & 0 & -H^*(\mathbf{x}) & 0 \\ 0 & 0 & 0 & -H^*(\mathbf{x}) \end{pmatrix},$$

$$\hat{\Gamma}^{(0)} = \hat{N} - \hat{I}\hat{N}\hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (20)$$

Introducing the matrix $\hat{\Sigma}(\mathbf{x},\mathrm{t};\ \mathbf{x}',\mathrm{t})$, we write (19) as follows:

$$\left[i\frac{\partial}{\partial t}\hat{E} - \hat{H}^{(0)}(\mathbf{x}) + \mu\hat{\Gamma}^{(0)}\right]\hat{G}(\mathbf{x},t;\ \mathbf{x}',t')$$
$$= \hat{E}\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')$$
$$+ \int d\mathbf{y}\,d\tau\,\hat{\Sigma}(\mathbf{x},\ t;\ \mathbf{y},\ \tau)\,\hat{G}(\mathbf{y},\ \tau;\ \mathbf{x}',t'). \quad (21)$$

In the absence of an external field in (21), we can transform to the Fourier components in x and t:

$$[\omega\hat{E} - \xi(k)\hat{\Gamma}^{(0)} - \hat{\Sigma}(k,\omega)]\,\hat{G}(k,\omega) = \hat{E}. \quad (22)$$

For the study of the structure of the self energy, we shall expand $\langle T\varphi(\mathbf{x},\mathrm{t})\,\psi_\gamma(\mathbf{x},\mathrm{t})\,\psi^\beta(\mathbf{x}',\mathrm{t}')\rangle$ in a perturbation-theory series. Since the vacuum averages in the interaction representation are expressed in terms of the matrix $\hat{G}^{(0)}$ by the usual rules, we can make use of the usual graphical tech-

niques. To each solid or dashed line, there corresponds an electron or phonon Green's function, multiplied by i, and to the simple vertex there corresponds the matrix $\hat{\Gamma}^{(0)}$ [see (20)]. We recall that by the electron Green's function we mean the matrix $\hat{G}^{(0)}$, defined by (17).

It is now clear that we can write down the Dyson equation which defines the connection of the self energy $\hat{\Sigma}$ with the exact Green's functions and the total vertex function $\hat{\Gamma}$:

$$\hat{\Sigma}(\mathbf{k}, \omega) = \frac{i}{(2\pi)^4} \hat{\Gamma}^{(0)} \int d\mathbf{k}_1 d\omega_1 \hat{G}(\mathbf{k}_1, \omega_1) \times D(\mathbf{k} - \mathbf{k}_1; \omega - \omega_1) \hat{\Gamma}(\mathbf{k}, \omega; \mathbf{k}_1, \omega_1).$$

To expand $\hat{\Gamma}$ in a perturbation theory series, we need to know the zeroth Green's functions which contain the (still undetermined) quantities $\tilde{\xi}(\mathbf{k})$ and $C(\mathbf{k})$. We shall assume that we are limited to a simple vertex $\hat{\Gamma}^{(0)}$. There is no necessity in the exact expressions for $\tilde{\xi}$ and C for an estimate of the contribution of higher approximations.

For a normal metal, as has already been pointed out, we can replace the vertex function by a simple vertex. Calculations, completely analogous to those performed by Migdal, lead to the conclusion that in our case we can, with the same accuracy, neglect all corrections and limit ourselves to a simple vertex. Without presenting the calculations here, we shall make only a few remarks. For computation of $\hat{\Sigma}$, the larger values of the transferred momentum $q \sim k_0$ are important. Therefore, for example, in a simple tripole, both electron Green's functions enter with a large difference in the argument, and are therefore shown to be almost orthogonal. The difference of the "superconducting" Green's function from the "normal" does not appear to be essential here. It would have been necessary to take into account the counter term of H' only in the calculation of subsequent corrections to the vertex function.

Thus, replacing $\hat{\Gamma}$ by $\hat{\Gamma}^{(0)}$, we obtain the following expression for $\hat{\Sigma}$:

$$\hat{\Sigma}(\mathbf{k}, \omega) = \frac{i}{(2\pi)^4} \hat{\Gamma}^{(0)} \int d\mathbf{k}_1 d\omega_1 \hat{G}(\mathbf{k}_1, \omega_1) \times D(\mathbf{k} - \mathbf{k}_1; \omega - \omega_1) \hat{\Gamma}^{(0)}. \quad (23)$$

The phonon Green's functions of the superconductor and the normal metal are practically identical, since the region of momenta which is important for the determination of D is large in comparison with the region of momenta close to the Fermi surface which is responsible for superconductivity. Therefore, we can use for D the expression of Migdal:[1]

$$D(q, \omega) = \alpha_q^2 \frac{\omega_q^{(0)}}{\omega_a} \left(\frac{1}{\omega - \omega_q + i\delta_1(q)} - \frac{1}{\omega + \omega_q - i\delta_1(q)} \right),$$

$$\omega_q^2 = (\omega_q^{(0)})^2 (1 - \lambda_0 + \lambda_0 q^2/8k_0^2),$$

$$\delta_1(q) = \pi\lambda_0 (\omega_q^{(0)})^2/4k_0 q. \quad (24)$$

Equations (22) and (23) now form a closed system. We eliminate Σ from the equations and write down the resultant matrix equation in the coordinate representation:

$$\left[i\frac{\partial}{\partial t}\hat{E} - \hat{H}^{(0)}(\mathbf{x}) + \mu\hat{\Gamma}^{(0)} \right] \hat{G}(\mathbf{x}, t, \mathbf{x}', t') = \hat{E}\delta(\mathbf{x} - \mathbf{x}')\delta(t - t') + i\int dy d\tau \hat{\Gamma}^{(0)} \hat{G}(\mathbf{x}, t; \mathbf{y}, \tau) \times D(\mathbf{x}, t; \mathbf{y}, \tau) \hat{\Gamma}^{(0)} \hat{G}(\mathbf{y}, \tau; \mathbf{x}', t'). \quad (25)$$

If we were to write out this equation in detail, then it would be seen that the system obtained here has the structure of the Gor'kiĭ equations. Equation (25) was derived for the case in which external fields are absent. However, it can be expected that it remains valid even when H(x) contains an external field, for example, a magnetic field. We note that it has a gauge invariant form.

We now turn our attention to Eqs. (22) and (23). The matrices $\hat{G}$ and $\hat{\Sigma}$ in them also have the same matrix structure as the matrix $\hat{G}^{(0)}$ [see (18)]. The formula for $\hat{G}$ is obtained from (18) if the index zero in the latter is omitted; $\tilde{\Sigma}$ is obtained from (18) if we replace $\hat{G}^{(0)}$ and $F^{(0)}$ by Σ_1 and Σ_2, respectively, where

$$\Sigma_1(\mathbf{k}, \omega) = \frac{i}{(2\pi)^4} \int d\mathbf{k}_1 d\omega_1 G(\mathbf{k}_1, \omega_1) D(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1), \quad (26a)$$

$$\Sigma_2(\mathbf{k}, \omega) = \frac{i}{(2\pi)^4} \int d\mathbf{k}_1 d\omega_1 F(\mathbf{k}_1, \omega_1) D(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1). \quad (26b)$$

It will be convenient for us to solve the equation for $\hat{\Sigma}$; therefore, we express $\hat{G}$ in terms of $\hat{\Sigma}$ by means of (22):

$$G = [\omega + \xi(k) + \Sigma_1(k, -\omega)] / \Omega(k, \omega),$$

$$F(k, \omega) = \Sigma_2(k, \omega) / \Omega(k, \omega),$$

$$\Omega(k, \omega) = [\omega - \xi(k) - \Sigma_1(k, \omega)] \times [\omega + \xi(k) + \Sigma_1(k_1 - \omega)] - [\Sigma_2(k, \omega)]^2. \quad (27)$$

We divide $\Sigma_1(\mathbf{k}, \omega)$ into an odd $f(\mathbf{k}, \omega)$ and an even $\mu_1(\mathbf{k}, \omega)$ part relative to ω. Then, by substituting in (26a) the expression for G from (27), we get the equations

$$f(\mathbf{k},\omega)=\frac{i}{(2\pi)^4}\int d\mathbf{k}_1 d\omega_1 \frac{\omega_1-f(\mathbf{k}_1,\omega_1)}{\Omega(\mathbf{k}_1\omega_1)} D(\mathbf{k}-\mathbf{k}_1,\omega-\omega_1), \quad (28)$$

$$\mu_1(\mathbf{k},\omega)=\frac{i}{(2\pi)^4}\int d\mathbf{k}_1 d\omega_1 \frac{\xi(\mathbf{k}_1)+\mu_1(\mathbf{k}_1,\omega_1)}{\Omega(\mathbf{k}_1,\omega_1)} \times D(\mathbf{k}-\mathbf{k}_1,\omega-\omega_1). \quad (29)$$

In Eq. (29), we divide the region of integration over $\mathbf{k}_1$ into two parts, with $|\xi(\mathbf{k}_1)|<\xi_1$ and $|\xi(\mathbf{k}_1)|>\xi_1$, respectively, where $\omega_0 \ll \xi_1 \ll E_F$. The integration over the first part vanishes because of the odd nature of the integrand relative to ξ (we can include the μ_1 in ξ under the integral). In the integral over the second part, we neglect Σ in comparison with ξ. We can then complete the integration over ω_1. Then, transforming from integration over the angles to integration over $q=|\mathbf{k}-\mathbf{k}_1|$, and taking it into account that in the important region, $\omega_q \ll |\xi|$, we obtain for μ_1 the expression

$$\mu_1(k,\omega)=-\frac{1}{8\pi^2 k}\int_0^{q_M} q\alpha_q^2 \frac{\omega_q^{(0)}}{\omega_q} dq \cdot P\int \frac{\xi(k_1)k_1}{\xi^2(k_1)-\omega^2} dk_1. \quad (30)$$

For $\omega \ll E_F$ and $k \sim k_0$, the function μ_1 does not depend on k and ω, and represents a correction to the chemical potential

$$\mu = E(k_0)+\mu_1.$$

Before proceeding to solve the equation for f and Σ_2, let us investigate certain analytical properties of the electron Green's function in a superconductor. We shall not undertake a rigorous derivation of these properties, and therefore shall consider for simplicity the Green's function of zero approximation (18). The function $G^{(0)}(k,\omega)$ can be written as

$$G^{(0)}(k,\omega)=(\omega+\xi(k))/(\omega-\varepsilon(k)+i\delta)(\omega+\varepsilon(k)-i\delta).$$

Simultaneously, we consider the function of complex argument

$$\tilde{G}^{(0)}(k,z)=(z+\xi(k))/[z-\varepsilon(k)][z+\varepsilon(k)],$$

which is analytic at all points in the z plane except the parts of the real axis $(-\infty,-\Delta)$ and $(\Delta,+\infty)$, where $\Delta=\epsilon(k_0)$ (Fig. 1). In the interval $(-\Delta,+\Delta)$, we have $\tilde{G}^{(0)}=G^{(0)}$. For $\omega \le -\Delta$, the value of $G^{(0)}$ coincides with $\tilde{G}^{(0)}$ on the lower side of the cut, while for $\omega \ge \Delta$, on the upper side of the cut. The values of $\tilde{G}^{(0)}$ on the opposite sides of the cuts are complex conjugates. Analytic continuation of $G^{(0)}$ in the upper half plane for $\omega<-\Delta$, and in the lower half plane for $\omega>\Delta$, has poles which determine the single particle spectrum of the system. Here, the path along which the analytic continuation follows must intersect the real axis outside the cut $(-\Delta,\Delta)$. Everything that has been said also applies to the function $F^{(0)}$. The exact function G and F must also possess the general properties that have been enumerated. For $\Delta=0$, these properties coincide with the analytical properties of the Green's function obtained in reference 8. Thus we can assume that the functions $\tilde{\Sigma}_1(k,z)$ and $\tilde{\Sigma}_2(k,z)$ exist and are analytic everywhere except for the parts of the real axis $(-\infty,-\Delta)$ and $(\Delta,+\infty)$ and coincide respectively with $\Sigma_1(k,\omega)$ and $\Sigma_2(k,\omega)$ on the lower side of the cut $(-\infty,-\Delta)$, at the cut of the real axis $(-\Delta,\Delta)$, and on the upper side of the cut $(\Delta,+\infty)$. The values of $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ on the upper side of the cut $(-\infty,-\Delta)$ and on the lower side of the cut $(\Delta,+\infty)$ are equal to Σ_1^* and Σ_2^*, respectively.

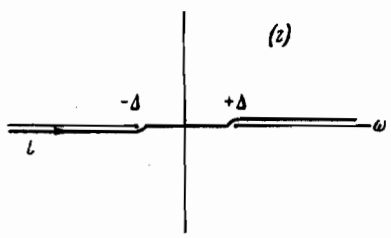

FIG. 1

We shall now employ these properties in the solution of the equations. We transform in (28) from integration over the real axis to integration over the contour L (Fig. 1), replacing the functions of real argument ω by the corresponding analytical functions. Moreover, we transform from integration over $\mathbf{k}_1$ to integration over ξ, $q=|\mathbf{k}-\mathbf{k}_1|$ and φ. Since $k \sim k_0$, and the integral over k_1 converges in the interval $k_1 \sim k_0$, then $dk_1 \approx q\,dq\,k_0^{-1}d\xi\,d\varphi$. As a result, f is shown not to be dependent on k. In what follows, we shall see that Σ_2 also does not depend on k. Therefore, we can write Eq. (28) in the form

$$f(\omega)=-\frac{i}{(2\pi)^3 k_0}\int_0^{q_1} q\alpha_q^2\frac{\omega_q^{(0)}}{\omega_q}dq\int_{-\infty}^{\infty}d\xi\int_L dz\,\frac{z-\tilde{f}(z)}{\xi^2-\tilde{\Omega}_1^2(z)} \times\left(\frac{1}{z-\omega-\omega_q+i\delta_1(q)}-\frac{1}{z-\omega+\omega_q-i\delta_1(q)}\right),$$

where $q_1=\min(2k_0, q_M)$ and the notation

$$\Omega_1^2(\omega)=[\omega-f(\omega)]^2-[\Sigma_2(\omega)]^2$$

has been introduced.

The function $[z-\omega-\omega_q+i\delta_1(q)]^{-1}$ is analytic in the upper half plane. Therefore, in the corresponding part of the integral, the contour L can be deformed into the contour L_1. By analogy, in this part of the integral, which contains $[z-\omega+\omega_q-i\delta_1(q)]^{-1}$, the contour L can be deformed into L_2 (Fig. 2). Further, since $[z-f(z)][\xi^2-\tilde{\Omega}_1^2(z)]^{-1}$ is an odd function, one can proceed from integration over L_1 to integration over L_2. Finally,

$$C(\omega)=\frac{1}{4\pi^2k_0}\int_0^{q_1}q\alpha_q^2\frac{\omega_q^{(0)}}{\omega_q}dq\int_0^{\infty}\frac{C(\omega_1(\xi))}{\sqrt{\xi^2+\Delta^2}}\frac{\omega_1(\xi)}{\omega_1^2(\xi)-\omega^2}d\xi. \quad (37b)$$

The result for C, obtained by Bogolyubov,[4] can be written in the following fashion:

$$C(\omega)=\frac{1}{2\pi^2k_0}\left[1-\frac{f_0(\omega)}{\omega}\right]^{-1}$$
$$\times\int_0^{q_1}q\alpha_q^2\frac{\omega_q^{(0)}}{\omega_q}dq\int_0^{\infty}\frac{C(\omega_1(\xi))}{\sqrt{\xi^2+\Delta^2}}\frac{d\xi}{\omega_1(\xi)+\omega+\omega_q}. \quad (38)$$

For $\omega \ll \omega_0$, our result does not differ from (38) and, in particular, the results for the gap Δ coincide. However, the behavior of $C(\omega)$ for $\omega \gtrsim \omega_0$ is different in the two cases. This difference, which does not exist for many problems, is connected with the following. The equation (38) is obtained from the principle of compensation of dangerous diagrams, which are actually dangerous only for $\omega \ll \omega_0$. Our formulas are derived from the Dyson equation, i.e., actually, are the summation of a definite class of diagrams, which exclude all "dangerous" diagrams. Therefore, (36) and (37) describe the behavior of $C(\omega)$ more exactly for large ω than does (38).

The imaginary part of $Q(\omega)$ is equal, for $\omega \ll \omega_0$, to

$$\operatorname{Im}Q(\omega)\equiv C_1(\omega)=\frac{1}{4\pi k_0}\frac{1}{1+\lambda}\int_0^{q_1}q\alpha_q^2\frac{\omega_q^{(0)}}{\omega_q}\frac{C(\omega-\omega_q)}{\sqrt{(\omega-\omega_q)^2-\Delta^2}}$$
$$\times\theta(\omega-\omega_q-\Delta)\,dq+\frac{1}{1+\lambda}\frac{f_1(\omega)}{\omega}C(\omega).$$

The principal terms of both components for $\alpha = (\omega-\Delta)/\Delta \ll 1$ are proportional to $\alpha^{5/2}$. They differ only in sign and cancel out. For thisis reason $C_1(\omega) \sim \alpha^{7/2}$, for $\alpha \ll 1$, and makes no contribution in practice to the imaginary part of the Green's function. For $\omega \gg \Delta$, the quantity C_1 does not play a role.

The spectrum and the damping of the excitations are determined by the roots of the equation

$$[\omega-f(\omega)]^2-[\xi(k)+\mu_1]-[\Sigma_2(\omega)]^2=0,$$

whence, for $\omega \ll \omega_0$,

$$\omega(k)=\varepsilon(k)-i\gamma(k),\quad \varepsilon(k)=\sqrt{\tilde{\xi}^2(k)+C^2(k)},$$
$$\gamma(k)=-f_1(\varepsilon(k))/(1+\lambda), \quad (39)$$

where $\tilde{\xi}=(\xi+\mu_1)/(1+\lambda)$ is the renormalized energy of the electron of the normal metal, reckoned from μ.

For $(\epsilon(k)-\Delta)/\Delta \ll 1$,

$$\gamma(k)=\frac{8}{15}\frac{\pi\lambda_0(2-\lambda_0)}{(1-\lambda_0)^2}\left(\frac{2\Delta}{\omega_0}\right)^2\left(\frac{\xi(k)}{2\Delta}\right)^5\Delta. \quad (40)$$

In the normal metal, the damping of the excitations in the vicinity of the Fermi surface falls off as ξ^3 for $\xi \gg \omega_0^2/E_F$ and as ξ^2 for $\xi \ll \omega_0^2/E_F$. The much faster decrease of $\gamma(k)$ for $k \to k_0$ is connected with the fact that the important part of the interaction of the electrons with phonons close to the Fermi surface is contained in the zero Hamiltonian.

The author thanks L. É. Gurevich for his numerous valuable suggestions and discussions.

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Translated by R. T. Beyer
183

$$C(\omega)=\frac{1}{4\pi^2k_0}\int_0^{q_1}q\alpha_q^2\frac{\omega_q^{(0)}}{\omega_q}dq\int_0^{\infty}\frac{C(\omega_1(\xi))}{\sqrt{\xi^2+\Delta^2}}\frac{\omega_1(\xi)}{\omega_1^2(\xi)-\omega^2}d\xi. \quad (37b)$$

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183